Tropical Abstractions of Max-Plus-Linear Systems

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1. Introduction

- The goal of this research is to generate the abstractions of Max-Plus-Linear (MPL) systems using tropical operations.
- Tropical operations are defined over $\mathbb{R}_{\max} = \mathbb{R} \cup \{\varepsilon := -\infty\}$,

$$a \oplus b := \max\{a, b\} \quad \text{and} \quad a \otimes b := a + b, \quad \forall a, b \in \mathbb{R}_{\max}$$

- The algebraic structure $(\mathbb{R}_{\max}, \oplus, \otimes)$ is well known as max-plus algebra (or tropical algebra).
- Inspired by the fact that an MPL system is a discrete-event model updating its state with operations in the tropical algebra.
- An autonomous MPL system is defined as $\mathbf{x}(k+1) = A \otimes x(k)$ where $A \in \mathbb{R}_{\max}^{n \times n}$ and $\mathbf{x}(k) \in \mathbb{R}^n$.

2. Abstractions of MPL systems

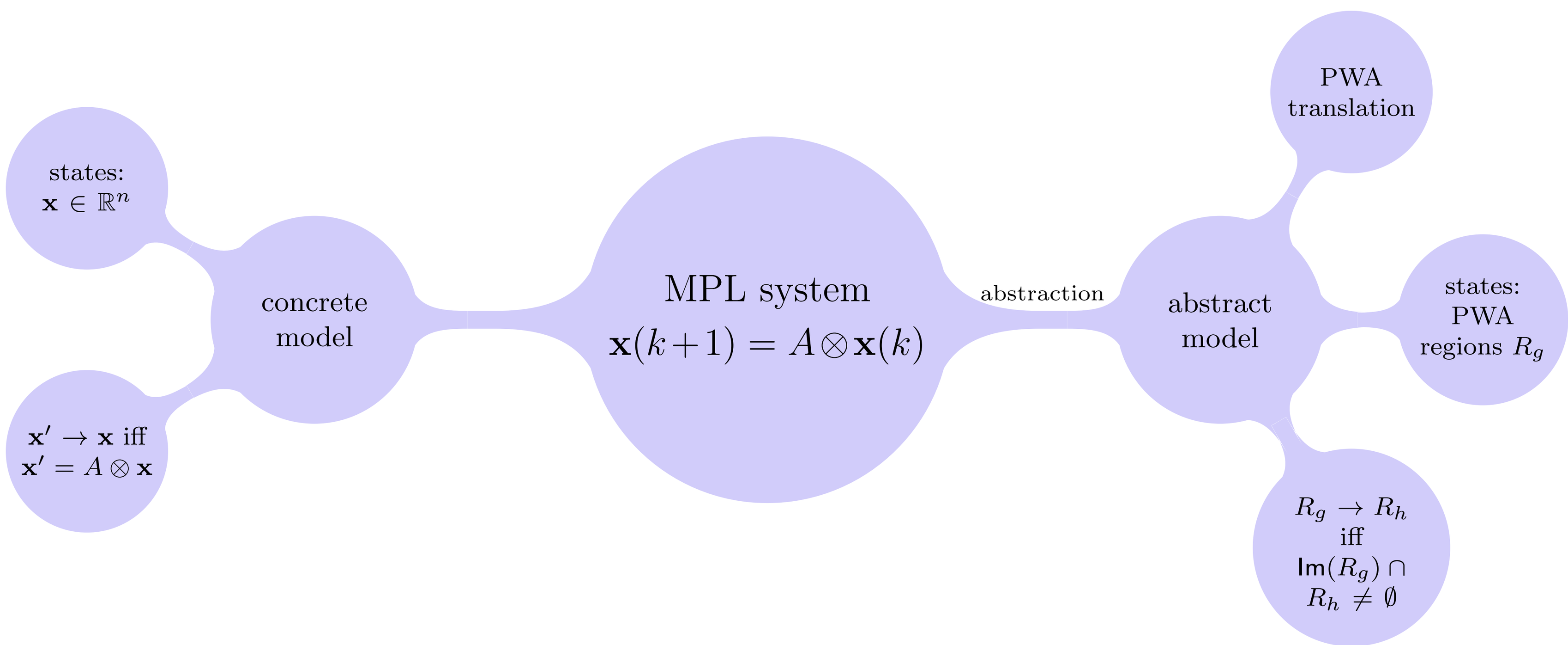

Figure 1: The framework of MPL system abstraction

- Abstraction of MPL system is PWA (Piecewise-Affine) system.
- PWA region is formulated as

$$R_g = \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} \{\mathbf{x} \in \mathbb{R}^n | x_{g_i} - x_j \geq A(i,j) - A(i,g_i)\},$$

where $g \in \{1, \ldots, n\}^n$ such that $A(i, g_i) \neq \varepsilon$ for all $1 \leq i \leq n$.
- R_g is a Difference-Bound Matrix (DBM).
- The corresponding affine dynamic is $\mathbf{x}' = A_g \otimes \mathbf{x}$ if $\mathbf{x} \in R_g$ where $A_g(i,j) = A(i,j)$ if $g_i = j$, and ε otherwise.
- Abstract states are defined as non-empty R_g after refinement.
- $\mathsf{Im}(R_g) = \{A_g \otimes \mathbf{x} \mid \mathbf{x} \in R_g\}$ and its dual $\mathsf{Inv}(R_g) = \{\mathbf{x} \mid A_g \otimes \mathbf{x} \in R_g\}$.

3. Tropical Abstractions of MPL systems

Difference-Bound Matrices

A DBM in $\mathbb{R}^n$ is the intersection of sets defined by $x_i - x_j \sim_{i,j} d_{i,j}$, where $\sim_{i,j} \in \{>, \geq\}$ and $d_{i,j} \in \mathbb{R} \cup \{-\infty\}$ for $0 \leq i, j \leq n$. The variable x_0 is set to be equal to 0.

- Each DBM can be expressed as a pair of matrices (D, S).
- The element $D(i,j)$ stores the bound variable $d_{i,j}$.
- S represents the *sign matrix* of the operator i.e. $S(i,j) = 1$ if $\sim_{i,j} = \geq$ and 0 otherwise.

Results

- Generating abstract states: $R_g = A_g^{\mathsf{c}} \otimes A \oplus I_n$.
  $A^{\mathsf{c}}(i,j) = -A(j,i)$ if $A(i,j) \neq \varepsilon$, and ε otherwise.
- Computing the image and inverse image of DBMs:

$$\mathsf{Im}(D) = A_g^{\mathsf{c}} \otimes D \otimes A_g \text{ and } \mathsf{Inv}(D) = A_g \otimes D \otimes A_g^{\mathsf{c}}$$

- DBM operations:
  Intersection: $D_1 \cap D_2 \equiv D_1 \oplus D_2$.
  Canonical-form: $\mathsf{cf}(D) = \bigoplus_{m=0}^{n+1} D^{\otimes m}$.

4. Numerical Benchmark

- Tropical abstractions versus VeriSiMPL 1.4 [2].
- For increasing n, we generate matrices $A \in \mathbb{R}_{\max}^{n \times n}$ with two finite elements in each row, with value ranging between 1 and 100.

Table 1: Generation of abstract states and transitions over 10 independent experiments. Each entry represents the average running time.

n	VeriSiMPL 1.4.		Tropical	
	time for generating abstract states	time for generating transitions	time for generating abstract states	time for generating transitions
3	7.51 ms	0.13 sec	4.04 ms	0.12 sec
4	11.29 ms	0.20 sec	5.23 ms	0.17 sec
5	18.51 ms	0.20 sec	5.16 ms	0.19 sec
6	49.22 ms	0.21 sec	9.99 ms	0.20 sec
7	90.88 ms	0.24 sec	15.88 ms	0.22 sec
8	0.21 sec	0.32 sec	0.04 sec	0.27 sec
9	0.52 sec	0.72 sec	0.07 sec	0.60 sec
10	1.25 sec	2.62 sec	0.14 sec	2.38 sec
11	3.87 sec	17.62 sec	0.35 sec	17.17 sec
12	8.34 sec	1.20 min	0.61 sec	1.10 min
13	26.17 sec	5.05 min	1.21 sec	4.98 min
14	1.81 min	41.14 min	0.06 min	40.61 min
15	10.29 min	2.63 hr	0.11 min	2.57 hr

5. Extensions

- Extension to non-autonomous MPL systems with dynamics that are characterised by non-square tropical matrices.
- Extension to predicate abstractions of MPL systems.
- Extension to other systems e.g. Min-Plus-Linear systems, Min-Max-Plus systems.

References

[1] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat. *Synchronization and linearity: an algebra for discrete event systems.* John Wiley & Sons Ltd, 1992.

[2] D. Adzkiya and A. Abate. VeriSiMPL: Verification via biSimulations of MPL models. *In Proc.* 10^{th} *International Conference on Quantitative Evaluation of Systems (QEST'13)*, 2013.

[3] D. Adzkiya, B. De Schutter, and A. Abate. Finite abstractions of max-plus-linear systems. *IEEE Transactions on Automatic Control*, 58(12):3039-3053, 2013.

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